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TUNED VIBRATION ABSORBERS WITH DRY FRICTION DAMPING

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SUMMARY

The response of a linear SDOF system subjected to harmonic excitation to which a Tuned Vibration Absorber with linear stiffness and dry friction damping is attached is considered. Based on an intuitive examination of the physical behaviour of the system, closed-form expressions for TMD optimum parameters and for the steady-state amplitudes of vibration are presented. Two examples allow the comparison between the predicted behaviour and that found by numerically integrating the equations of motion. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: passive control; tuned mass dampers; friction damping

INTRODUCTION

In both Civil and Mechanical Engineering applications, the Tuned Vibration Absorber or Tuned Mass Damper (TMD) is widely recognized to be an effective means to reduce the response of inadequately damped systems. Its fields of application range from small rotating machinery to large Civil Engineering structures subjected to earthquake or wind loading, and consequently its characteristics can be quite different from one case to the other.

The physical principles underlying its behaviour, however, are quite simple, and common to all different applications. Its purpose is, in fact, that of reducing the resonant component of the response by adding a force in opposite phase with respect to the excitation, and is achieved by 'tuning' the additional vibrating mass to a frequency close to the resonant frequency of the primary system. The effect is that of apparently increasing the damping in the primary system. In fact, if the response of a primary system to which a TMD has been added is equated to that of a SDOF system, an effective value of the damping of the SDOF system is calculated much higher than the inherent damping of the primary system.

The first study of the behaviour of a system to which a TMD has been attached was published by Ormondroyd and Den Hartog.[1] The response of an undamped linear SDOF system subjected

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to a sinusoidal excitation with either an undamped or a viscously damped linear vibration absorber is considered. Including some additions brought by Brock,[2] the complete theory can be found in the book by Den Hartog.[3]

From the study it emerges that, though quite effective in reducing the response of the primary system at resonance (it in fact cancels the motion of the primary system), the undamped damper has no practical applicability. First because the amplitude of vibration of the secondary system is infinite, independently of the value of the added mass. Second because its effect of eliminating the original resonance of the primary system is accompanied by the introduction of two new undamped resonances at frequencies slightly apart from the original. This effect is undesired when the frequency of the excitation has some degree of uncertainty.

On the other hand, the performance of a viscously damped vibration absorber was analysed with the so-called *fixed points theory* developed by Ormondroyd and Den Hartog. The theory relies on very intuitive observations on the changes brought to the primary system standard response curve by the addition of a secondary vibrating mass. Considering the mass of the secondary system as a given parameter, closed-form expression for the tuning (i.e. the ratio between the natural frequencies of the secondary and primary systems respectively) and for the TMD's damping ratio are derived that minimize the primary system's maximum response with varying frequency of the excitation.

The *fixed points theory* was extended by Snowdon[4] to systems having complex stiffness. The extension was aimed at analysing the performance of the TMD in reducing the response of rubber-like material mounted equipment.

Many papers appeared since then dealing with TMDs, either considering more complicated mechanical systems or more complicated forms of excitation. However, with the exception of a few, they appear to be bound to the hypothesis of a linear behaviour of both the primary and secondary system.

On the other hand, in many applications, a non-linear behaviour has to be taken into account, for two reasons. First because in some cases the primary or the secondary system exhibit a non-linear behaviour that cannot be neglected. Second because in some other cases a non-linear behaviour of the secondary system can be welcome if it results in a better performance of the damping device or in a simplification in terms of construction and maintenance.

A TMD in which the damping is provided by two friction devices acting at a right angle with the direction of motion of the secondary mass, was proposed by Inaudi and Kelly.[5] The system is non-linear and exhibits a hysteretic behaviour. With the hypothesis of small deformations of the damping device, a linearization technique was used to derive expressions for the optimum parameters under a white noise excitation. The proposed device shows a level of effectiveness comparable to that of the viscous damper.

Again a damper having a hysteretic behaviour was considered by Abe.[6] It was observed in fact that a linear viscous damper attached to a structure having an hysteretic behaviour loses its effectiveness due to the change occurring in the natural frequency of structure with large amplitude motion. In the paper a structure with a bilinear hysteresis is considered, to which a TMD with a similar behaviour is attached. A procedure is indicated for the optimization of the TMD parameters based on the equivalent linearization, and the better effectiveness of the device shown through a numerical example.

In this paper the performance of a TMD having linear stiffness dry friction damping is considered, when attached to a damped linear SDOF system subjected to harmonic excitation.

The analyses are carried out based on the understanding of the physical behaviour of the system, and bring to closed-form expressions for the optimum TMD parameters and system response. Two examples allow the comparison between the results coming from the theory and the results of the numerical integration of the equations of motion, and point out the cases in which a friction TMD is more effective than a viscous TMD.

STEADY-STATE RESPONSE OF A SDOF SYSTEM WITH A FRICTION TMD TO A SINUSOIDAL EXCITATION

Effects of the amplitude and frequency of the excitation on the response of the system

Consider the 2DOF system of Figure 1, in which m_1 represents the primary system and m_2 represents the TMD. The primary system is provided with a linear stiffness k_1 and a viscous damping c_1, while the secondary system has a linear stiffness k_2 and a dry friction damping c_0.

With the usual notation the equations of motion are

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_0\,\mathrm{sgn}(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = q(t) \quad \text{(1a)}$$

$$m_2\ddot{x}_2 + c_0\,\mathrm{sgn}(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \quad \text{(1b)}$$

Equations (1) can be rewritten as

$$\ddot{x}_1 + 2\beta_1\omega_1\dot{x}_1 + \frac{c_0}{m_1}\,\mathrm{sgn}(\dot{x}_1 - \dot{x}_2) + \omega_1^2x_1 + \Omega^2\omega_1^2\mu(x_1 - x_2) = \frac{1}{m_1}\,q(t) \quad \text{(2a)}$$

$$\ddot{x}_2 + \frac{c_0}{\mu m_1}\,\mathrm{sgn}(\dot{x}_2 - \dot{x}_1) + \Omega^2\omega_1^2(x_2 - x_1) = 0 \quad \text{(2b)}$$

where

$$\mu = \frac{m_2}{m_1} \quad \text{(3a)}$$

$$\Omega = \frac{\omega_2}{\omega_1} \quad \text{(3b)}$$

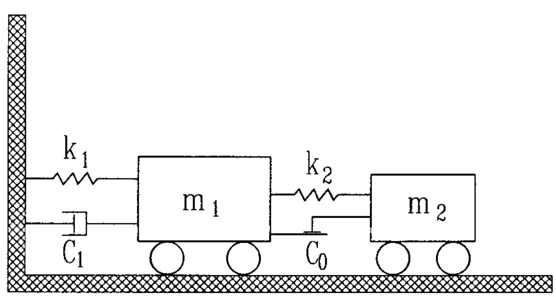

Figure 1. SDOF system with a friction Tuned Mass Damper

are the mass ratio and the tuning ratio respectively, and where ω_1 and β_1 are the circular frequency and damping ratio of the primary system respectively, and ω_2 is the circular frequency of the TMD.

In this paragraph the analysis of the steady-state response of the system described by equations (2) and subjected to a sinusoidal excitation:

$$q(t) = q_0 \sin \omega t \tag{4}$$

will be attempted using a physical argumentation.

It should be clearly understood from the beginning that the behaviour of the system shown in Figure 1 is discontinuous. For given values of the system properties, in fact, a threshold value of the amplitude of the excitation exists under which no relative motion occurs between the primary and secondary systems (stick behaviour). This is because the inertia force acting on the secondary system is lower than the friction force, and thus unable to start a relative motion. Under these circumstances, the TMD acts as dead mass, and globally the system behaves as a linear SDOF system, which implies a sinusoidal response.

The only effect of the presence of the TMD is in this case that of reducing the resonant frequency of the primary system to the value

$$\omega_1^* = \frac{\omega_1}{\sqrt{1+\mu}} \tag{5}$$

The amplitudes of oscillation $\hat{x}_1$ and $\hat{x}_2$ of the two masses can be calculated as that of a linear SDOF system, and are

$$\hat{x}_1(\varpi) = \hat{x}_2(\varpi) = \frac{q_0}{k_1} \frac{1}{[(1-\varpi^2(1+\mu))^2 + 4\beta_1^2\varpi^2(1+\mu)]^{0.5}} \tag{6}$$

where ϖ is the ratio between the frequency of the excitation and the natural frequency of the primary system.

On the other hand, the steady-state amplitude $\hat{x}_{1\ell}$ of oscillation of the primary system at which a relative motion starts can be calculated equating the maximum inertia force acting on the TMD in a sinusoidal motion to the friction force:

$$m_2 \hat{\ddot{x}}_1 = c_0 \;\; \Rightarrow \;\; \hat{x}_{1\ell} = \frac{c_0}{\mu m_1 \omega^2} \tag{7}$$

which is dependent on the friction force, on the added mass and on the frequency of the excitation.

Equating equations (6) and (7) and solving for q_0 the amplitude of the excitation at which the relative motion starts can be calculated:

$$q_{0\ell} = \frac{c_0}{\mu \varpi^2} [(1-\varpi^2(1+\mu))^2 + 4\beta_1^2\varpi^2(1+\mu)]^{0.5} \tag{8}$$

For amplitudes of the exciting force larger than the limit value given by equation (8), a relative motion occurs between the primary and the secondary systems (stick-slip behaviour). If the amplitude of the excitation is, however, only slightly higher than the threshold value of equation (8), the relative motion lasts for very short amounts of time, i.e. only when the acceleration of the primary system exceeds the ratio between the friction force and the mass of the secondary

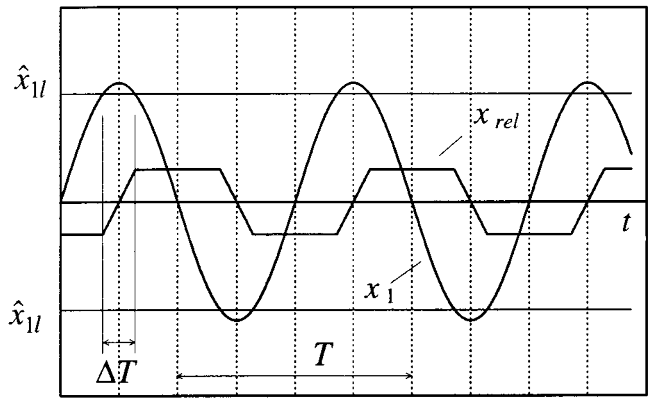

Figure 2. Motion of the primary system and relative motion for an amplitude of the excitation slightly exceeding the limit value of equation (8)

system, i.e. the maximum acceleration that can be applied to the secondary system through the friction connection.¶ The relative motion is for this reason that of a SDOF system acted upon by a pulse force in phase with the motion of the primary system and is characterized by steps 90° out of phase with respect to the motion of the primary system, independently of the phase of the primary system (Figure 2). In this case if the inertia force of the primary system is large compared to the friction force, the motion of the primary system is not affected by the non-linearity of the damper, and thus remains almost sinusoidal, otherwise it is disturbed by the presence of the non-linear secondary system.

As the amplitude of the excitation further increases, the amplitude of vibration of the two masses increase and the inertia and elastic forces acting on the two masses increase as well while the friction force stays the same. This implies that the non-linearity of the system reduces as the amplitude of the excitation increases. Based on this observation it seems reasonable to assume that as the amplitude of the excitation tends to infinity, the motion tends to be sinusoidal again, i.e.

$$x_1(t) = \hat{x}_1 \sin(\omega t + \varphi_1) \tag{9a}$$

$$x_2(t) = \hat{x}_2 \sin(\omega t + \varphi_2) \tag{9b}$$

From the preceding observations three different behaviours can be pointed out with varying amplitude of the excitation:

1. for small values of the excitation the system behaves as a lightly damped linear SDOF system, with a reduced natural frequency with respect to that of the plain primary system;
2. for intermediate values of the excitation the system behaves as a well damped non-linear 2DOF system;

¶ The elastic force applied on the mass m_2 by the spring k_2 is in this case is almost negligible due to the small relative displacements.

3. for high values of the excitation the system behaves as a lightly damped 2DOF, tending to a linear behaviour as the excitation goes to infinity.

Moreover, from equation (8) it can be noticed that the limit value of the amplitude of the excitation depends on the frequency of the excitation. For this reason for a given value of the amplitude of the excitation q_0 a range of frequencies of the excitation exists in which the damper is effective (i.e. there is relative motion), and outside which the system behaves as a SDOF system (no relative motion). The upper and lower bounds of this range (depending on q_0) can be calculated from equation (8), and, for small values of the damping β_1 of the primary system, turn out to be

$$\varpi_\ell^2 = \left[1 + \left(1 \pm \frac{q_0}{c_0}\right)\mu\right]^{-1} \tag{10}$$

Limit behaviour of the system under large excitation

Under the assumption that the motion of the system is described by equation (9), it is possible to calculate the work done over one cycle by the forces acting on the secondary system, for the displacement of the secondary system. The work done by the elastic force is

$$W_k^{(2)} = \int_0^T k_2 \hat{x}_{\mathrm{rel}} \sin(\omega t + \varphi_{\mathrm{rel}}) \omega \hat{x}_2 \cos(\omega t + \varphi_2)\, \mathrm{d}t = \pi k_2 \hat{x}_{\mathrm{rel}} \hat{x}_1 \sin(\varphi_{\mathrm{rel}} - \varphi_1) \tag{11}$$

and the work done by the friction force is

$$W_c^{(2)} = \int_0^T c_0 \,\mathrm{sgn}(\omega t + \varphi_{\mathrm{rel}}) \omega \hat{x}_2 \cos(\omega t + \varphi_2)\, \mathrm{d}t = 4 c_0 \hat{x}_{\mathrm{rel}} + 4 c_0 \hat{x}_1 \cos(\varphi_{\mathrm{rel}} - \varphi_1) \tag{12}$$

where $\hat{x}_{\mathrm{rel}}$ and φ_{rel} are the amplitude and the phase of the relative motion.

For the energy balance the total work has to be zero, i.e.

$$\pi k_2 \hat{x}_{\mathrm{rel}} \hat{x}_1 \sin(\varphi_{\mathrm{rel}} - \varphi_1) + 4 c_0 \hat{x}_{\mathrm{rel}} + 4 c_0 \hat{x}_1 \cos(\varphi_{\mathrm{rel}} - \varphi_1) = 0 \tag{13}$$

It was pointed out that for low values of the excitation the system behaves as a SDOF system, and due to its low damping, the phase ($\varphi_1 = \varphi_2$) can have any value between $-\pi$ and 0 with changing frequency of the excitation. However, as the amplitude of the excitation increases a relative motion between the primary system and the TMD starts always 90° out of phase with respect to the motion of the primary system, independently of the frequency of the excitation (i.e. independently of the phase of the primary system). Due to the relative motion the TMD applies to the primary system an elastic force in-phase with the relative motion, and a force due to friction in-phase with the relative speed.

As the amplitude of the excitation further increases the relative motion increases, but this leads to an increase only in the magnitude of the elastic force that the TMD applies to the primary system. Like in a linear system, when the system is at resonance the presence of the damper is equivalent to an increase in the damping of the primary system. However, the amount of additional damping brought to the system by the damper is in this case a function of the amplitude of vibration, and decreases as the latter increases.

Despite of this complication an upper bound for the amplitude of vibration of the primary system at resonance can be simply calculated from equation (13). In fact, at resonance, as $q_0 \to \infty$, $\varphi_1 \to -\pi/2$ and $\varphi_{\mathrm{rel}} \to -\pi$, and from equation (13):

$$\hat{x}_{1\mathrm{u}} = \frac{4c_0}{\pi\mu m_1 \omega_1^2 \Omega^2} = \frac{4}{\pi}\frac{1}{\Omega^2}\,\hat{x}_{1\ell} \tag{14}$$

This result shows how narrow the range of the amplitude of vibration of the primary system for which the secondary system is effective is. In other words, it is found that the secondary system limits the amplitude of vibration of the primary system at resonance to $1{\cdot}273/\Omega^2$ times the amplitude at which the secondary system starts its motion.

The corresponding relative amplitude of vibration can be calculated through the energy balance of the 2DOF system. The dissipation over one cycle of the viscous damping of the primary system and of the friction damping of the secondary system are, respectively,

$$W_\beta = \int_0^T 2\beta_1\omega_1 m_1 \omega^2 \hat{x}_1^2 \cos^2(\omega t + \varphi_1)\,\mathrm{d}t = 2\pi\beta_1\omega_1 m_1 \omega \hat{x}_1^2 \tag{15}$$

$$W_c = \int_0^T c_0 \omega \hat{x}_{\mathrm{rel}} |\cos(\omega t + \varphi_{\mathrm{rel}})|\,\mathrm{d}t = 4c_0 \hat{x}_{\mathrm{rel}} \tag{16}$$

and the work done by the external force is

$$W_q = \int_0^T q_0 \sin\omega t\, \omega \hat{x}_1 \cos(\omega t + \varphi_1)\,\mathrm{d}t = -\pi q_0 \hat{x}_1 \sin\varphi_1 \tag{17}$$

Setting the total work equal to zero one finds

$$\hat{x}_{\mathrm{rel}} = -\frac{\pi}{4c_0}(q_0 \sin\varphi_1 + 2\beta_1\omega_1 m_1 \omega \hat{x}_1)\hat{x}_1 \tag{18}$$

Upon substitution of equation (14) into equation (18) an expression for the relative amplitude of oscillation at resonance can be found:

$$\hat{x}_{\mathrm{rel}} = \frac{1}{\mu\Omega^2}\left(\frac{q_0}{k_1} - \frac{8}{\pi}\frac{c_0}{\mu m_1}\beta_1 \frac{1}{\omega_1^2\Omega^2}\right) \tag{19}$$

Equations (14) and (19) prove to give an accurate description of the behaviour at resonance, and are in agreement with the theory of the undamped damper by Ormondroyd and Den Hartog. In fact, it appears that as the TMD's damping goes to zero the motion of the primary system goes to zero and the relative motion to a finite value given by equation (19), coincident with that found, using a different approach, for the undamped damper.

However when the system is not at resonance they are quite misleading, and they tend to underestimate the system response. This is primarily due to two facts. The first is that they do not take into account the change occurring in the phases as the frequency of the excitation changes. The second is that they do not allow for the resonant behaviour of the lightly damped 2DOF system.

On the other hand, the resonance of the 2DOF system resulting from the addition of a friction TMD to a SDOF is a condition to be avoided more than described. In the remaining part of this paragraph, closed-form expression of the TMD's parameters will be derived that prevent the 2DOF system resonance from occurring.

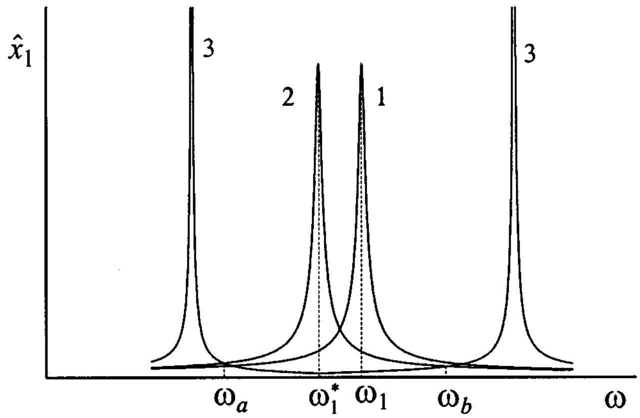

Figure 3. Response of the system with and without TMD

Optimization of the TMD's parameters

In Figure 3 the changes brought to the system response by the addition of a friction TMD are qualitatively shown. Curve #1 shows the response of the plain SDOF system. Curve #2 represents the response of the system to which a friction TMD has been added, in the case in which the friction force is larger than the maximum inertia force acting on the secondary system, i.e. in the case in which no relative motion occurs between the damper and the main mass. As pointed out the system still behaves as an SDOF, but shows a lower resonant frequency. This has to be considered a first limit behaviour for the system. A second limit behaviour is that of curve #3, showing the response of the system to which a friction TMD has been added, in the case in which the ratio between the friction force and the amplitude of the excitation goes to zero; this is the case of the undamped TMD. It is clear that a range of frequencies of the excitation exists in which the response of the modified SDOF of curve #2 is larger than that of the 2DOF system of curve #3, and outside which the response of the SDOF system is lower than that of the 2DOF.

The upper and lower bounds of this range can be calculated by equating the response of a SDOF system with an undamped damper (curve #3), given by the expression

$$\hat{x}_1(\varpi) = \frac{q_0}{k_1} \frac{\left| 1 - \dfrac{\varpi^2}{\Omega^2} \right|}{\left| \left(1 - \dfrac{\varpi^2}{\Omega^2}\right)(1 + \mu\Omega^2 - \varpi^2) - \mu\Omega^2 \right|} \tag{20}$$

to that of the modified SDOF system described by equation (6) (curve #2).

For low values of the damping in the primary system the two non-dimensional limit frequencies ϖ_a and ϖ_b are given by the following expression:

$$\varpi_a^2, \varpi_b^2 = \frac{(1 + \mu)\Omega^2 + 1 \mp \sqrt{(1 + \mu)^2\Omega^4 + 1 - 2\Omega^2}}{2 + \mu} \tag{21}$$

It is clear at this point that for frequencies of the excitation in the range given by equation (21) the motion is effectively controlled by the damping device, while outside that range the resonant behaviour of the 2DOF system controls the motion.

DESIGN CRITERIA FOR FRICTION TMDs

The results of the previous paragraph allow the design of a friction tuned mass damper for a system subjected to sinusoidal excitation. Two cases will be considered: firstly the case in which the excitation has a well prescribed frequency, corresponding to the resonant frequency of the system to be damped, then the case in which the excitation is expected to span a range of frequencies. In the second case the value of the mass ratio is chosen *a priori* based on practical reasons, and expressions for the optimum values of the tuning and friction force are derived.

System at resonance

When the system is at resonance the best performance of the damping device is obtained when this is tuned to the natural frequency of the primary system. Once the maximum allowed displacement of the primary and secondary system are prescribed, equations (14) and (19) can be used to calculate the required auxiliary mass and the corresponding value of the friction force.

Solving equation (14) for c_0 and substituting into equation (19) one obtains

$$\mu = \frac{1}{\hat{x}_{\mathrm{rel}}}\left(\frac{q_0}{k_1} - 2\beta_1 \hat{x}_{1\mathrm{u}}\right) \tag{22}$$

that gives the required auxiliary mass.

Equation (14) is then used to calculate the required friction force:

$$c_0 = \frac{\pi}{4}\, m_1 \omega_1^2 \mu \hat{x}_{1\mathrm{u}} \tag{23}$$

in which μ is the value derived from equation (22).

System subjected to an excitation with varying frequency

The above procedure is quite straightforward and leads to very accurate results (see the numerical examples in the next paragraph). However, the case in which the excitation is at a known frequency is hard to find in real life, since in most cases a range of frequencies of excitation has to be considered. In this second case the design of the damping device can be carried out as follows.

In the previous paragraph it was pointed out that the system response is mitigated by the presence of the TMD only when the frequency of the excitation is in the range given by equation (21), while it is amplified outside. However, the undesired amplification of the response can be prevented if the damper's parameters (tuning and friction force) are chosen so as to make the damper effective only inside the mentioned range. In other words, one can choose the TMD parameters in a way that the response follows curve #3 of Figure 3 as long as this is below curve #2, and follows curve #2 otherwise.

Equating equations (10) and (21) one obtains the two values of the friction force that make the minimum and maximum frequencies ϖ_ℓ of effectiveness of the TMD coincide with the two limit frequencies ϖ_a and ϖ_b of the system. The resulting values of the friction force, that will be designated as optimum values, are

$$c_0^{\text{opt}} = \mp\, q_0 \mu \frac{(1+\mu)\Omega^2 + 1 \mp \sqrt{(1+\mu)^2\Omega^4 + 1 - 2\Omega^2}}{(1+\mu)^2\Omega^2 - 1 \mp (1+\mu)\sqrt{(1+\mu)^2\Omega^4 + 1 - 2\Omega^2}} \tag{24}$$

where the upper signs come from matching the first values in equations (10) and (12) and the lower signs from matching the second values.

The two values of the friction force given by equation (24) are generally not coincident. However, a proper choice of the tuning will make them coincide. This criterion is used to choose the optimum tuning.

Equating the two values of equation (24) one obtains

$$\Omega^{\text{opt}} = \frac{1}{\sqrt{1+\mu}} \tag{25}$$

The required friction force can then be calculated upon substitution of equations (25) in equation (24).

Equation (25) shows that the best performance of the damper is obtained when this is tuned to the frequency of what has been called the modified SDOF system (see equation (5)). Moreover, it can be seen that the optimum frequency of a friction TMD is higher than that of a viscous TMD for which $\Omega^{\text{opt}} = 1/(1+\mu)$.

The response of the system at resonance can be calculated from equations (14) and (19), but this is not necessarily the maximum response the system shows with varying frequency of the excitation. The maximum response can either occur at resonance or when the frequency of the excitation matches the first of the two values given by equation (21), depending on the system parameters.

The response in the latter case can be calculated substituting equation (25) in equation (21), and the result in equation (6). For small values of the damping of the primary system it is

$$\hat{x}_1 = \frac{q_0}{k_1} \frac{\mu + 2}{-\mu + \sqrt{2\mu(\mu+1)}} \tag{26}$$

The corresponding amplitude of vibration of the secondary system can then be calculated through equation (18).

If the calculated amplitudes of vibration are acceptable then the design can be considered satisfactory. A large value of the amplitude of vibration of the secondary system indicates that a larger added mass is required. A small value of the amplitude of vibration of the secondary system is a symptom of an overdesign of the damping device.

NUMERICAL STUDIES

Some numerical experiments were carried out in order to validate the results of the preceding paragraph. For this purpose the SIMULINK software was used. In Figure 4 the flow diagram of

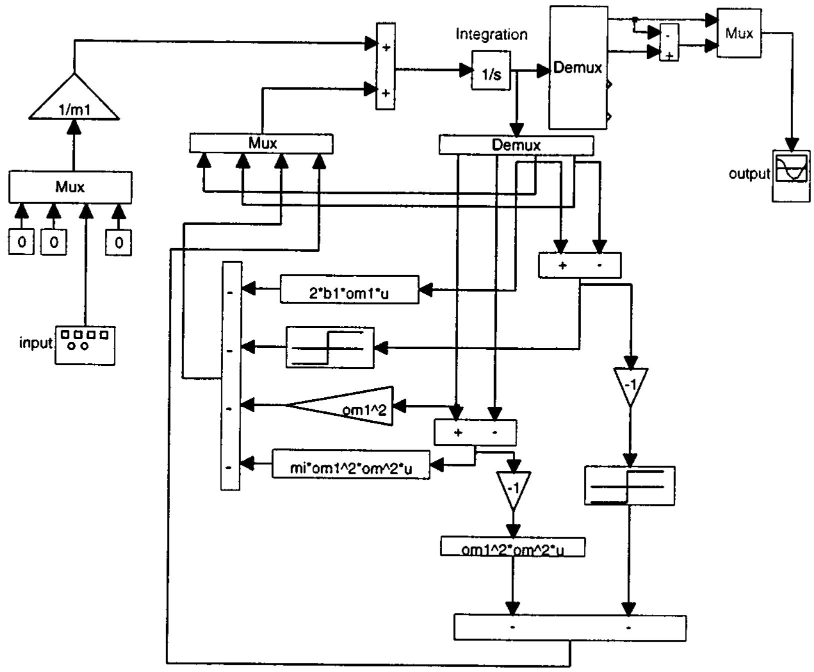

Figure 4. Flow diagram of routine for the integration of the equation of motion

the routine used is shown. The integration was carried out using a Runge-Kutta algorithm with variable time step. The dry friction was modelled as $\mathrm{sgn}(\dot{x}_{\mathrm{rel}})$.

Example 1: System at resonance

As a first example a system having a mass $m = 1000$ kg, a natural frequency $f = 5$ Hz and a damping $\beta_1 = 0{\cdot}005$ is considered at resonance. The maximum amplitude of the exciting force is considered to be $q_0 = 1000$ N. The amplitude of the response at resonance would be $\hat{x}_{10} = 101$ mm.

In order to reduce the resonant response to a maximum value of 2 mm a friction TMD is attached to the primary system. The amplitude of vibration of the secondary system is required to be limited to 20 mm.

From equation (22) it appears that a mass of 15 per cent that of the primary system has to be added, and from equation (23) a value of the friction force $c_0 = 77{\cdot}5$ N is calculated.

The result of the numerical experiment is an amplitude of vibration of the primary system $\hat{x}_1 = 2{\cdot}01$ mm and a relative amplitude of vibration $\hat{x}_{\mathrm{rel}} = 19{\cdot}5$ mm, in very good agreement with the predictions.

Moreover, in order to verify the behaviour of the system at resonance, pointed out in the preceding paragraphs, the equations of motion were integrated for different values of the amplitude of the excitation.

Figures 5–7 show the results of the simulations. In Figure 5 the response of the secondary system is plotted as a function of the response of the primary system. It can be clearly seen that the

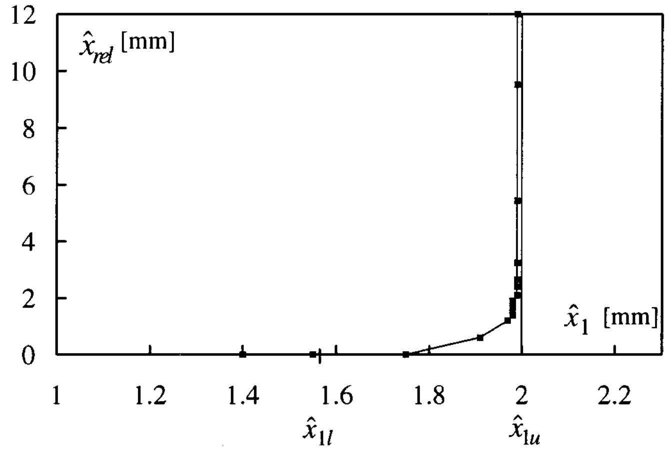

Figure 5. Example 1: relative amplitude of oscillation vs. amplitude of oscillation of the primary system

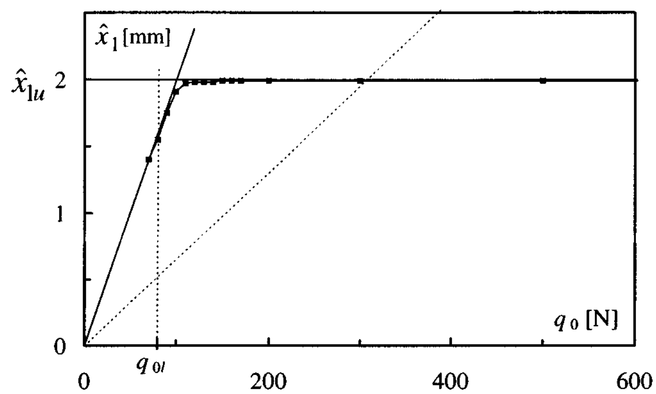

Figure 6. Example 1: amplitude of oscillation of the primary system vs. amplitude of the excitation

value $\hat{x}_{1u}$ represents an asymptote for the response of the primary system. In Figure 6 the response of the primary system is plotted as a function of the excitation. The solid line through the origin is the response of the plain primary system, the horizontal line indicates the predicted upper limit of the response of the primary system and the dashed line through the origin represents the response in the case in which an optimum viscous TMD is attached to the system. It appears that the viscous TMD is more effective for amplitudes of the excitation not exceeding $q_0 = 300$ N, while for larger values of the excitation the friction TMD becomes more effective. In Figure 7 the relative amplitude of vibration is plotted as a function of the excitation, together with equation (19). Equation (19) proves to be very accurate in the prediction of the relative amplitude of vibration when the motion of the secondary system is larger than that of the primary system. However when the amplitudes of vibration of the two systems are comparable, the assumption that the motion is sinusoidal is no more acceptable, and equation (19) tends to overestimate the response of the secondary system.

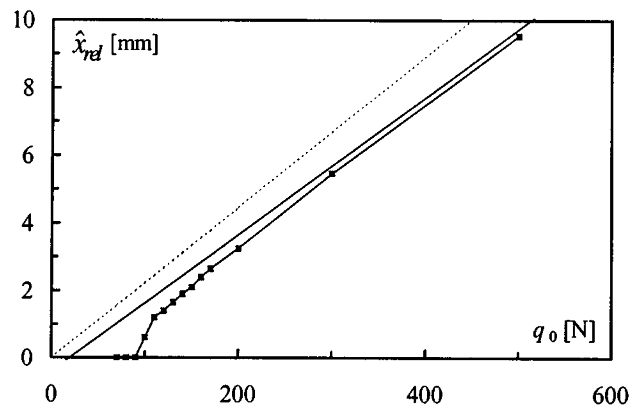

Figure 7. Example 1: relative amplitude of oscillation vs. amplitude of the excitation

The agreement between the numerical and the theoretical results seems to be good. The three different behaviours of the system anticipated earlier in the paper were found from the numerical investigation, i.e.

1. for $< q_{0\ell}$ no relative motion occurs between the primary and the secondary system, the motion is sinusoidal and the amplitude of oscillation is that of a linear system;
2. for intermediate values of the excitation the behaviour is strongly non-linear, and cannot be easily investigated;
3. for $q_0 \to \infty$ the motion is again sinusoidal, the amplitude of oscillation of the primary system tends to a limit value and that of the secondary system can be calculated by energy balance.

Example 2: System subjected to an excitation with varying frequency

The same system considered in the previous example was again taken into consideration in the case in which it is subjected to an excitation of prescribed amplitude and varying frequency.

The amplitude of the excitation is again $q_0 = 1000$ N.

The optimum tuning and the required friction force are calculated from equations (25) and (24) respectively: $\Omega^{\mathrm{opt}} = 0{\cdot}976$, $c_0 = 309$ N.

The amplitudes of oscillation of the system at resonance are calculated from equations (14) and (19) and are $\hat{x}_{1\mathrm{u}} = 8{\cdot}37$ mm and $\hat{x}_{\mathrm{rel}} = 19{\cdot}52$ mm, while from equation (26) an amplitude of vibration of the primary system $\hat{x}_1 = 7{\cdot}58$ mm is calculated, lower than the corresponding value at resonance.

The maximum amplitudes of vibration resulting from the numerical experiments are $\hat{x}_1 = 8{\cdot}8$ mm and $\hat{x}_{\mathrm{rel}} = 19{\cdot}2$ mm respectively, both occurring at frequencies lower than the resonant. The agreement with the predicted values seems good.

In Figure 8 the amplitude of vibration of the primary system is plotted versus the non-dimensional frequency of the excitation ϖ. The effectiveness of the TMD in the range of

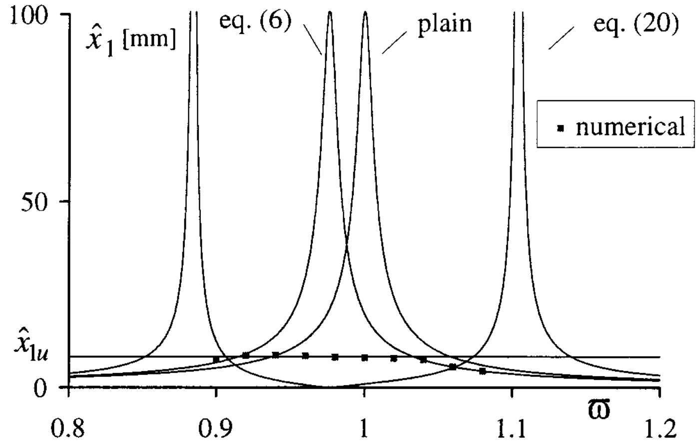

Figure 8. Example 2: amplitude of vibration of the primary system vs. non-dimensional frequency of excitation

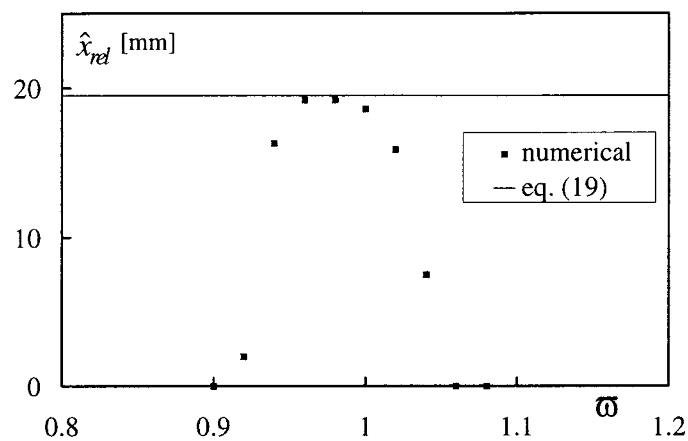

Figure 9. Example 2: relative amplitude of vibration vs. non-dimensional frequency of excitation

frequencies comprised between the two limit values given of equation (21) is evident. In Figure 9 the amplitude of vibration of the secondary system is plotted as a function of ϖ. From this plot also the range of effectiveness of the damping device can be read.

In order to investigate the sensitivity to tuning, the calculations were repeated for $\Omega = 1$. In this case the required friction force resulting from equation (24) is $c_0 = 370$ N, and the predicted maximum amplitude of vibration resulting from equations (14) and (19) are $\hat{x}_1 = 9{\cdot}54$ mm and $\hat{x}_{\mathrm{rel}} = 18{\cdot}35$ mm respectively. The corresponding values resulting from numerical experiments are $\hat{x}_1 = 10{\cdot}6$ mm and $\hat{x}_{\mathrm{rel}} = 18{\cdot}7$ mm, respectively. It appears that by optimizing the tuning a 17%

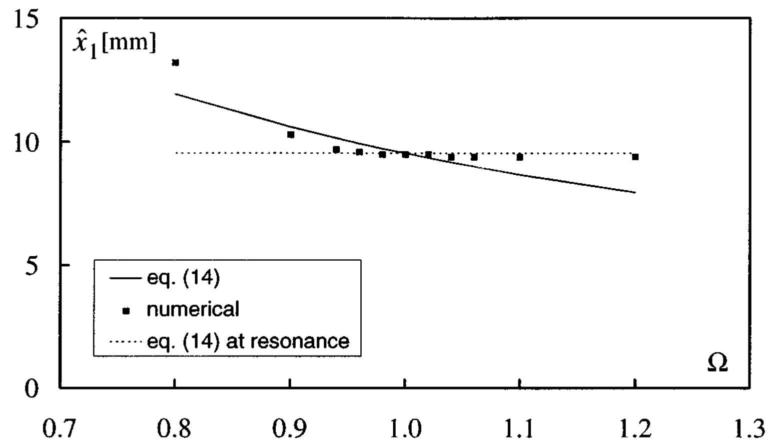

Figure 10. Example 2: amplitude of vibration of the primary system vs. tuning ratio

reduction in the maximum response of the primary system is achieved, with a slight increase in the response of the secondary system.

If compared to the viscous TMD the friction TMD has in this case a worse performance. At the optimum tuning $\Omega = 0{\cdot}952$, the viscous damper brings in fact to a maximum amplitude of vibration of the primary system $\hat{x}_1 = 6{\cdot}49$ mm, 26 per cent lower than that of the optimum friction damper. On the other hand, the relative amplitude of vibration of the viscous TMD would be $\hat{x}_{rel} = 22{\cdot}3$ mm, 10 per cent higher than that of the friction TMD.

Finally, in order to assess the accuracy of equation (14) in predicting the maximum response of the primary system when the damper is tuned to a frequency different from that given by equation (25), the equations of motion were numerically integrated with $q_0 = 1000$ N and $\varpi = 1$, and for a friction force of 370 N, assuming different values of the tuning ratio. The results of the numerical experiments are shown in Figure 10, where the amplitude of vibration of the primary system is plotted as a function of the tuning ratio. Together with the experimental results equation (14) is plotted, showing how this well predicts the response for values of the tuning lower than one, but tends to underestimate the response for higher values of the tuning. For higher values of the tuning it appears in fact that a more accurate prediction of the response is given by the value that equation (14) assumes for a unitary tuning.

CONCLUSIONS

The response of a linear SDOF system subjected to harmonic excitation to which a TMD with linear stiffness and dry friction damping is attached was considered. Some intuitive observations were used to understand the behaviour of the system and to derive closed-form expressions for the optimum tuning and for the optimum friction force as well as for the steady-state amplitudes of vibration of the main system and of the additional mass.

The theory is based on the observation that in spite of the strong non-linearity, as the amplitude of vibration becomes large the system tends to behave as a lightly damped linear system. A consistent choice of the secondary system parameters allows, however, to avoid the resonant behaviour of the latter, bringing to a standard response curve almost flat in a broad range of frequencies.

The reliability of the proposed expressions was tested by comparing the theoretical results to the results of the numerical integration of the equations of motions. In spite of the simplifications introduced the comparison shows a very good agreement between the theory and numerical results.

Moreover, the comparison between the performances of the viscous and the friction dampers shows that, though less effective for low values of the excitation, the friction damper tends to be more effective as the amplitude of the excitation becomes large.

APPENDIX: NOTATION

c_0	friction force
c_0^{opt}	optimum friction force
c_1	viscous damping of the primary system
k_1, k_2	primary and secondary system stiffness
m_1, m_2	primary and secondary system mass
$q(t)$	exciting force
q_0	amplitude of the harmonic excitation
$q_{0\ell}$	limit amplitude of the excitation
W_β, W_c	dissipation over one cycle due to the viscous and friction damping
W_q	work of the excitation over one cycle
$W_k^{(2)}, W_c^{(2)}$	work over one cycle of the secondary spring and friction damper for the displacement of the secondary system
$x_1(t), x_2(t)$	primary and secondary system displacement
$x_{1\ell}$	lower limit amplitude of oscillation of the primary system
$\hat{x}_{1\mathrm{u}}$	upper limit amplitude of oscillation of the primary system
$\hat{x}_1, \hat{x}_2$	amplitudes of oscillation of the primary and secondary system
$\hat{x}_{\mathrm{rel}}$	amplitudes of the relative motion
β_1	damping ratio of the primary system
φ_1, φ_2	primary and secondary system phases
φ_{rel}	phase of the relative motion
μ	mass ratio
ω	circular frequency of the excitation
ϖ	non-dimensional frequency of the excitation
ω_1, ω_2	primary and secondary system natural circular frequencies
ω_1^*	circular frequency of the modified SDOF system
ϖ_a, ϖ_b	non-dimensional limit frequency of the 2DOF system
ϖ_e	non-dimensional limit frequency of effectiveness of the damper
Ω	tuning ratio
Ω^{opt}	optimum tuning ratio

REFERENCES

1. J. Ormondroyd and J. P. Den Hartog, 'The theory of the dynamic vibration absorber', *J. Appl. Mech. ASME* **50**, 9–22 (1928).
2. J. E. Brock, 'A note on the damped vibration absorber', *J. Appl. Mech. ASME* **13** (A-284), (1946).
3. J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill, New York, 1956.
4. J. C. Snowdon, 'Steady-state behavior of the dynamic absorber', *J. Acoust. Soc. Am.* **31**, 1096–1103 (1959).
5. J. A. Inaudi and J. M. Kelly, 'Mass damper using friction-dissipating devices', *J. Engng. Mech. ASCE* **121**, 142–149 (1995).
6. M. Abe, 'Tuned mass dampers for structures with bilinear hysteresis', *J. Engng. Mech. ASCE* **122**, 797–800 (1996).

